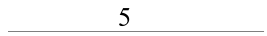
A Concise Introduction to Vectors

By Daniel D'Agostino, March 2010

Vectors and Scalars

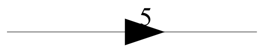
A **scalar quantity** is a quantity consisting of **magnitude only**.

e.g. time: 5 seconds

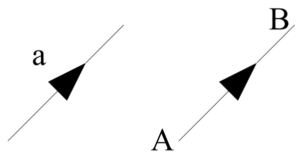


A **vector quantity**, on the other hand, is made up of **magnitude** as well as **direction**.

e.g. displacement: 5km to the east

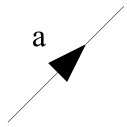


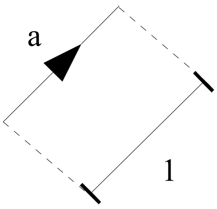
A vector is represented by a directed straight line and is denoted by either a single letter (e.g. **a**) or a connection between two points (e.g. $\overrightarrow{AB}$).



Vector Measurement

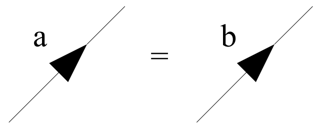
The **modulus** of a vector **a** is its length, and is denoted by |**a**|.



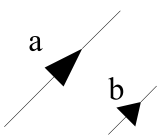


A **unit vector**, denoted by $\hat{a}$ is a vector of **length 1** (|**a**| = 1).

Two vectors are **equal** if they have the **same magnitude** and **same direction**.



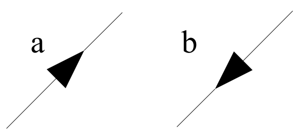
If vectors are **parallel**, they have the same direction. Since only the magnitude varies, parallel vectors are multiples of each other.



a = λ**b**

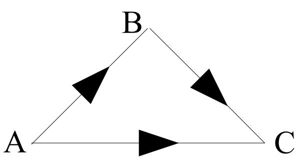
If **a** is parallel to **b** then **a** = λ**b**, where λ is a scalar ($\lambda \in \mathbb{R}$).

It follows that if λ is negative, then **a** and **b** have opposite directions.



a = -**b**

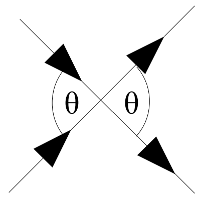
Addition of Vectors



If A, B and C are considered to be cities, then $\overrightarrow{AC}$ as well as $\overrightarrow{AB}+\overrightarrow{BC}$ are both valid ways of reaching C from A.

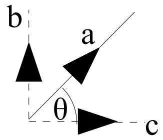
$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$$

The Angle between two Vectors



The angle between two vectors is measured as the angle between where two vectors converge or diverge.

Resolution of Vectors by Components

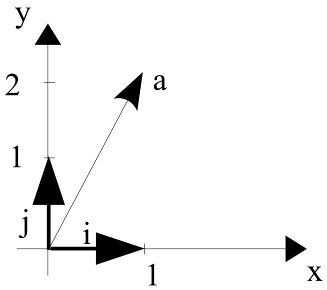


It follows from the vector addition rule that a vector **a** can be broken down into two perpendicular components. Note that the location of a vector in space does not matter; the aside diagram means that **a** = **b** + **c**. From trigonometry:

$$Vertical\,component: b = a\sin(\theta)$$

$$Vertical\,component: c = a\cos(\theta)$$

Vector Coordinates in 2D Space

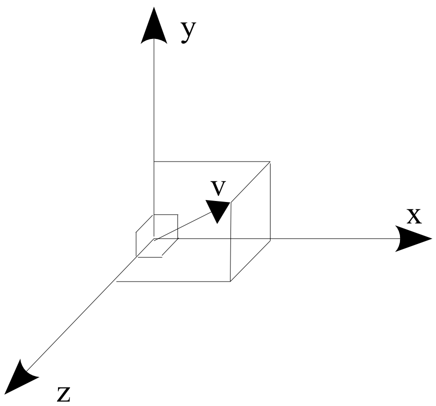


If **i** and **j** are **unit vectors** on the x- and y-axis respectively, this allows us to express any point in 2D space using these unit vectors. For example, a point (1,2) in space is expressed as

a = **i** + 2**j** or $\begin{pmatrix}1\\2\end{pmatrix}$

This is called a **position vector**.

Vector Coordinates in 3D Space

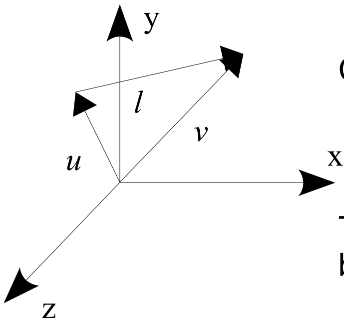


The concept of vector positioning in 2D can be extended to the third dimension by the addition of a **k** unit vector to handle positioning on the z-axis.

v = a**i** + b**j** + c**k** or $\begin{pmatrix}1\\2\\3\end{pmatrix}$

Also, $|\underline{\boldsymbol{v}}| = \sqrt{a^2 + b^2 + c^2}$

Vectors joining two points

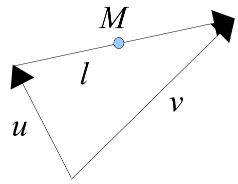


Consider two vectors: $\underline{\mathbf{u}} = x_1\underline{\mathbf{i}} + y_1\underline{\mathbf{j}} + z_1\underline{\mathbf{k}}$

$\underline{\mathbf{v}} = x_2\underline{\mathbf{i}} + y_2\underline{\mathbf{j}} + z_2\underline{\mathbf{k}}$

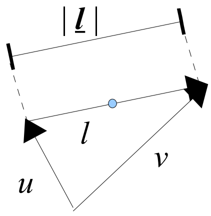
The **vector representing the line** joining the two points is given by:

$$\underline{\boldsymbol{l}} = \underline{\boldsymbol{v}} - \underline{\boldsymbol{u}} = (x_2 - x_1)\underline{\boldsymbol{i}} + (y_2 - y_1)\underline{\boldsymbol{j}} + (z_2 - z_1)\underline{\boldsymbol{k}}$$



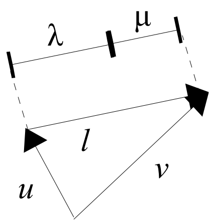
The **length of the line** joining the two points is given by:

$$|\underline{\boldsymbol{l}}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



The **midpoint of the line** joining two points is given by:

$$M = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2)\right)$$

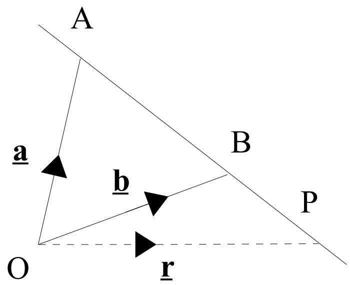


The position vector of a **point dividing a line in a given ratio** is given by:

$$r = \frac{\mu \underline{\boldsymbol{a}} + \lambda \underline{\boldsymbol{b}}}{\lambda + \mu}$$

Equation of a line in 3D

Vector equation



Consider a line through two known fixed points A and B.

The vector equation of the line is given by the locus of points in the direction of $\overrightarrow{AB}$ passing through A. If P is a point on the line with position vector $\underline{\mathbf{r}}$, then

$\underline{\mathbf{r}} = \lambda \overrightarrow{AB}$ (parallel vectors)

$\underline{\mathbf{r}} = \lambda (\underline{\mathbf{b}} - \underline{\mathbf{a}})$ ($\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ = $\underline{\mathbf{b}} - \underline{\mathbf{a}}$)

$\underline{\mathbf{r}} - \underline{\mathbf{a}} = \lambda (\underline{\mathbf{b}} - \underline{\mathbf{a}})$ ($\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$ = $\underline{\mathbf{r}} - \underline{\mathbf{a}}$)

$$\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda(\underline{\boldsymbol{b}} - \underline{\boldsymbol{a}})$$

Note: λ is a parameter whose value determines the location of a point on the line.

Parametric equation

Let **r** = x**i** + y**j** + z**k**

a = x_1**i** + y_1**j** + z_1**k**

b = a**i** + b**j** + c**k** where **b** is parallel to the line

As before, **r** = **a** + λ **b**

Substituting **r**, **a** and **b**: x**i** + y**j** + z**k** = x_1**i** + y_1**j** + z_1**k** + λ (a**i** + b**j** + c**k**)

Comparing coefficients of **I**, **j** and **k**:

$$x = x_1 + a\lambda$$
$$y = y_1 + b\lambda$$
$$z = z_1 + c\lambda$$

Cartesian equation

By rearranging the parametric equations with λ subject:

$$\lambda = \left(\frac{x - x_1}{a}\right); \ \lambda = \left(\frac{y - y_1}{b}\right); \ \lambda = \left(\frac{z - z_1}{c}\right)$$

Thence;

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Direction Ratios

In a vector **v** = a**i** + b**j** + c**k**, the ratios of *a*:*b*:*c* are called the **direction ratios** of **v**.

Pairs of Lines

In 2D, lines are either parallel, or they intersect at some point.

In 3D, 2 lines can be:

1. parallel – the two lines have the **same direction ratios**
2. not parallel and intersecting
3. skew (not parallel and not intersecting)

When working with a pair of lines such as

$$\underline{r_1} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k} + \lambda(x_1\underline{i} + y_1\underline{j} + z_1\underline{k})$$

$$\underline{r_2} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k} + \mu(x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$$

the following algorithm is followed:

1. Check if the lines are parallel
   a) Check the direction ratios of the direction vectors (i.e. the ones with the parameters λ or μ). If they are equal, then the lines are parallel.
   b) If they are not equal, then the lines either intersect or are skew.
      1. Equate the line equations.
      2. Compare coefficients of **i**, **j** and **k** to obtain three equations in terms of λ and μ.
      3. Use two of the equations to find the values of λ and μ.
      4. Substitute the values of λ and μ in the third equation.
         a) If LHS = RHS, then the lines intersecting.
         b) If LHS ≠ RHS, then the lines are skew.

The Dot or Scalar Product

The dot product is one way of multiplying two vectors that gives a resulting scalar value.

If **a** and **b** are two vectors, and θ is the angle between them, then

$$\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$$

This is useful in finding a resultant magnitude of the two vectors (e.g. the resultant of two forces), but also in determining the nature of the angle between two vectors (see below).

Properties of the dot product

1. Angle properties
   a) Same-direction parallel vectors: $\underline{a}.\underline{b} = |a||b|$ (since cos 0° = 1)
   b) Opposite-direction parallel vectors: $\underline{a}.\underline{b} = -|a||b|$ (since cos 180° = -1)
   c) Perpendicular vectors: $\underline{a}.\underline{b} = 0$ (since cos 90° = -1)
   d) Equal vectors: $\underline{a}.\underline{a} = |a|^2$ (since cos 0° = 1 and $|a|.|a| = |a|^2$)
2. Operator properties:
   a) Commutativity: $\underline{a}.\underline{b} = \underline{b}.\underline{a}$
   b) Distributivity across addition: $\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a} + \underline{c}$